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On Singularities of Power-Series

P. SZÜSZ

Department of Mathematics, SUNY, Stony Brook, New York 11794

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A classical theorem of Hadamard ([1], see also Titchmarsh [2]) states the following: Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

be a power-series with radius of convergence 1. Further, suppose that

$$a_k = 0 \text{ if } k \neq k_n ,$$

where $k_1, k_2, \ldots$ is a subsequence of 1, 2, 3,... satisfying the condition

$$\frac{k_{n+1}}{k_n} \geqslant \theta > 1. \tag{1}$$

Then the circle $|z| = 1$ is the natural boundary of $f(z)$.

There are several proofs and generalizations of this result. I mention Fabry's gap theorem replacing (1) by the weaker condition

$$\frac{k_n}{n} \to \infty \text{ as } n \to \infty. \tag{2}$$

For a proof see Landau [3], p. 76. It is known that (2) cannot be replaced by the still weaker condition

$$\overline{\lim}\,(k_{n+1} - k_n) = \infty; \tag{3}$$

see L. Ilieff [4], p. 3. On the other hand, one can ask whether (2) can be replaced by a weaker condition while imposing conditions on the non-vanishing coefficients. Results in this direction have been obtained by R. P. Boas [5] and H. Claus [6].

In a recent paper [7] I gave a proof of Hadamard's gap theorem based only on Stirlings formula. Now I prove a "finite form" of a gap theorem. The main result of the present paper is the following.

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THEOREM 1. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power-series with radius of convergence 1. Suppose that*

$$a_k = O(k^m). \tag{4}$$

Denote by $a_k(\delta)$ the kth coefficient of the power series of $f(z)$ about the point δ, where $0 < \delta < 1$. Suppose that there is a subsequence $k_1, k_2, \ldots$ of the sequence of natural numbers such that

$$\lim_{n\to\infty} | a_{k_n} |^{1/k_n} = 1 \tag{5}$$

and with some $\epsilon > 0$

$$| a_{(1-\delta_k)k}(\delta_k)| > \epsilon a_k \binom{k}{(1-\delta_k)k} \delta_k^{k\delta_k}, \tag{6}$$

δ_k being a number satisfying $k\delta_k = M_k$ (M_k is a natural number) and

$$M_k \geqslant \frac{1}{| a_k |^4} \geqslant M. \tag{7}$$

Then the arc of the unit circle e^{it} with

$$| t | < c_1 \left(\frac{\log M - \log \epsilon}{M}\right)^{1/2} \tag{8}$$

contains at least one singularity of $f(z)$; here c_1 is an absolute constant.

It is easy to see that if there are "sufficiently long" gaps in the power-series $f(z)$, then a dominance of the type (6) will occur. The dominance of type (6) with arbitrary large M assumes that $z = 1$ is a singular point.

As an application of Theorem 1, I show that for $f(z)$ whose coefficients satisfy (4), and (7) the condition (3) assures non-continuability. Further I give a new proof of Szegö's theorem [8], according to which a power-series whose coefficients take only a finite set of values, is either a rational function of a special kind or cannot be continued beyond $| z | = 1$. Section 1 contains the proof of Theorem 1, Section 2 some applications.

1. Proof of Theorem 1

Since the singular points of $f(z)$ and

$$\sum_{k=0}^{\infty} \frac{a_k}{k(k+1)\cdots(k+m+2)} z^{k+m+2} = \frac{1}{(m+2)!} \int_0^z f(t)(z-t)^{m+2}\, dt$$

coincide, we may suppose without loss of generality that

$$|f(z)| \leqslant 1 \qquad |z| < 1.$$

First, I prove

LEMMA 1.1. *Let $f(z)$ be any function regular in $|z| < 1$ and satisfying 1.1. Suppose that $f(z)$ can be continued beyond $|z| = 1$ into a domain containing an arc of the length 2ρ around 1, that is, into a domain containing the numbers $e^{it}(|t| \leqslant \rho)$. Then, writing*

$$f(z) = \sum_{k=0}^{\infty} a_k(\delta)(z-\delta)^k \; (0 \leqslant \delta \leqslant 1) \tag{1.2}$$

we have for $0 < \delta < \epsilon$

$$\left| \frac{1}{|a_k(\delta)|^{1/k}} - (1-\delta) \right| \frac{1}{\delta} > 1 - \cos\rho, \tag{1.3}$$

where ϵ depends only on the domain into which $f(z)$ can be analytically continued.

Proof. Without loss of generality we may suppose that (1, 1) holds in a domain containing the arc e^{it}, $(|t| \leqslant \rho)$. Then, if δ is so small that $f(z)$ is analytic in $|z-\delta| \leqslant ((\cos\rho - \delta)^2 + \sin^2\rho)^{1/2} = ((1-\delta)^2 + 2\delta(1-\cos\rho)^{1/2}$ and there it satisfies (1.1), we have

$$a_k(\delta) = \frac{1}{2\pi i} \int_c \frac{f(\varphi)}{(\varphi-\delta)^{k+1}}\, d\varphi, \tag{1.4}$$

where c is the circle $|\varphi - \delta| = ((1-\delta)^2 + 2\delta(1-\cos\rho)$. Hence

$$|a_k(\delta)|^{1/k} < \left[(1-\delta)\left(1 + \frac{2\theta}{1-\delta}(1-\cos\rho)\right)^{1/2}\right]^{-1}$$

which proves (1.3).

Now we are able to finish our proof. Let $f(z)$ be a function satisfying the conditions of Theorem 1.

Let (here and in the sequel) k denote a natural number belonging to the subsequence satisfying (5), (6), and (7).

Then we have

$$|a_{(1-\delta_k)k}(\delta_k)| = \left| \sum_{l \geqslant (1-\delta_k)k} \binom{l}{(1-\delta_k)\,k} \delta_k^{l-(1-\delta_k)k} a_l \right|$$

$$\geqslant \epsilon a_k \binom{k}{(1-\delta_k)\,k} \delta_k^{\delta_k k}$$

Estimating the term $\binom{k}{(1-\delta_k)k}\delta_k^{\delta_k k}$ by Stirling's formula (which is possible if $\delta_k k = M$, where M "large" but fixed) we obtain

$$|a_{(1-\delta_k)k}(\delta_k)| \geqslant \epsilon a_k (1-\delta_k)^{-(1-\delta_k)k} \frac{1}{(2\pi)^{1/2}} \frac{1}{(1-\delta_k)^{1/2}} \cdot \frac{1}{(M)^{1/2}}$$

Hence

$$|a_{(1-\delta_k)k}(\delta_k)|^{-1/(1-\delta_k)k} < (1-\delta_k) \exp\Big(-\frac{1}{(1-\delta_k)k}(\log \epsilon + \log a_k - \tfrac{1}{2}\log 2\pi(1-\delta_k)M\Big),$$

or

$$\Big|\,|a_{(1-\delta_k)k}(\delta)|^{-1/(1-\delta_k)k} - (1-\delta_k)\Big|\frac{1}{\delta_k}$$
$$\leqslant (1-\delta_k)\Big(\frac{\log M}{4k} - \frac{\log \epsilon}{k} + \frac{\log 2\pi(1-\delta_k)}{2k} + O\Big(\frac{1}{k^2}\Big)\Big)\frac{1}{\delta_k}. \qquad (1.5)$$

Expressing δ_k by (7) we obtain for the right-hand side of (1.5) the upper bound

$$\frac{\log M}{4M} - \frac{\log \epsilon}{M} + \frac{K}{M} + o(1).$$

By Lemma 1.1 Theorem 1 follows.

2. Applications

First, I prove a gap-theorem

THEOREM 2. *Suppose that $a_l = 0(1)$ and with an infinity of $k - s$*

$$|a_k| > \epsilon \max_{l \neq k} |a_l|, \qquad (2.1)$$

further that

$$a_l = 0 \qquad \text{if} \quad 0 < |l-k| \leqslant N. \qquad (2.2)$$

Then any arc of the unit-circle, whose length is greater than

$$c_2(\epsilon)\frac{\log^{1/2} N}{N^{2/3}} \qquad (2.3)$$

contains a singularity of $f(z)$.

Remarks (1) The restrictions $a_l = O(1)$ and (2.1) could be replaced by weaker ones at the expense of some calculations; for instance $a_l = O(1)$ could be replaced by $a_l = O(l)$ and (2.2) by

$$a_l = 0 \qquad \text{if} \quad 0 < l - k < N. \tag{2.2'}$$

Since the idea of the proof is clearer in the present form and the present form is sufficient for a further application, I confine myself to this simpler form.

(2) Theorem 2 is a result similar to the results of H. Claus [5], but not contained in them.

Proof. I have to show the existence of $\delta_k - s$ satisfying (6) and

$$k\delta = N. \tag{2.4}$$

To this end *I* use probability theory. Let δ be a number $0 < \delta < 1$, which will be determined later. Further let ξ be a random variable with

$$P(\xi = m) = (1 - \delta)^{(1-\delta)k} \binom{(1-\delta)\,k + m}{(1-\delta)\,k} \delta^m (m = 0, 1, \ldots,) \tag{2.5}$$

(where δ is chosen such that $k\delta$ is a natural number)

First *I* calculate the expectation $E(\xi)$ and variance $D^2(\xi)$. An easy calculation gives

$$E(\xi) = \delta k \tag{2.6}$$

$$D^2(\xi) = \frac{\delta k}{1 - \delta} \tag{2.7}$$

Čebyshev's inequality, applied to ξ yields

$$\sum_{|m - \delta k| > \lambda(k\delta/(1-\delta))^{1/2}} \binom{(1-\delta)\,k + m}{(1-\delta)\,k} \delta^m < \lambda^{-2}(1-\delta)^{-(1-\delta)k},$$

or, putting $\lambda = N((1 - \delta)/\delta k)^{1/2}$

$$\sum_{|m-\delta k| > N} \binom{(1-\delta)\,k + m}{(1-\delta)\,k} \delta^m < \frac{\delta k}{N^2(1-\delta)} (1-\delta)^{-(1-\delta)k} \tag{2.8}$$

On the other hand, we have by Stirling's formula,

$$\binom{k}{(1-\delta)\,k} \delta^{k\delta} \sim \frac{1}{(1-\delta)^{(1-\delta)k}} \frac{1}{(2\pi)^{1/2}} \frac{1}{(1-\delta)^{1/2}} \frac{1}{(k\delta)^{1/2}} \cdot \tag{2.9}$$

Therefore if $\delta k = \epsilon' N^{z/3}$ with some sufficiently small ϵ' depending only on ϵ of (2.1), then (6) and all the assumptions of Theorem 1 are satisfied. Therefore the interval

$$e^{it}, \mid t \mid < c_2(\epsilon) \frac{(\log N)^{1/2}}{N^{2/3}}$$

contains a singular point. Since $f(z)$, and also $f(e^{i\psi}z)$, satisfies the conditions (2.1) and (2.2), any arc of the unit circle of the length $c_2(\epsilon)[(\log N^{1/2}/N^{2/3}]$ contains a singular point, which proves our Theorem 2.

As an application of Theorem 2 *I* give a new proof of the following theorem.

THEOREM 3 (*G. Szego* [8] *see also Duffin and Schaeffer* [9]). *Let $f(z)$ be a power-series whose coefficients take only a finite set of values. Then either $f(z) = \pi(z)/(1 - z^m)$, where $\pi(z)$ is a polynomial or $f(z)$ cannot be continued beyond $\mid z \mid = 1$.*

Proof. Let d_1, d_2 ... d_ν be the values which can be taken. Then the number of all N tuples which can be taken is

$$\nu^N.$$

Denote by A_{Nn} the N-tuple $(a_n, a_{n+1} \cdots a_{n+N-1})$ and by $D_1 \cdots D_{\nu N}$ its possible values. Since there are ν^N values for the $A_{N,n}$ in any interval $(n, n + \nu^N)$, there must be at least one D_j which is taken by two different $A_{N,n}$. By the pigeon-hole principle either there is ρ, $0 < \rho \leqslant \nu^N$, such that

$$A_{N,n} = A_{N,n+\rho}, \tag{2.10}$$

or there are an infinity of n such that (2.10) holds but

$$A_{N+1,n} \neq A_{N+1,n+\rho}.$$

Then

$$f_1(z) = (1 - z^\rho) f(z) = \sum_{l=0}^{\infty} (a_l - a_{l-\rho}) z^l = \sum_{l=0}^{\infty} a_l^{(*)} z^l$$

has an infinity of gaps of length N, and $f_1(z) \neq \pi(z)$. Now using the same argument again we obtain the existence of a polynomial $\pi(z)$ of degree $\leqslant N/2$ and of

$$f_2(z) = \pi(z) f_1(z) = \sum_{l=0}^{\infty} a_l^{(**)} z^l,$$

for which there is an infinity of $k - s$ k_1, $k_2 \cdots$ such that

$$a_l = O(1) \tag{2.11}$$

$$a_{k_n} \geqslant c, \tag{2.11}$$

and

$$a_l = O \text{ for } O < | l - k | < \frac{N}{2};$$

therefore by Theorem 2 any arc of $|z| = 1$ of length at least c (log $N)/N^{2/3})^{1/2}$ contains a singularity of $f_2(z)$, that is, of $f(z)$. Since N can be taken arbitrarily large, Szegö's theorem follows.

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